Math 254-1 Exam 0 Solutions

1. Carefully state the definition of "linear combination". Give two examples.

   A linear combination of some variables is those variables, each multiplied by any constant, added together. Many examples are possible: $3x + 7y$, $8x + 0y + 2z$, 0.

2. Carefully state the definition of "subspace". Give two examples.

   A subspace is a vector space whose vectors are contained in another vector space. Many examples are possible: $\{(0,0)\}$ is a subspace of $\mathbb{R}^2$, $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$, $\{(a,0) : a \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

3. Consider the vector space $\mathbb{R}^3$. Determine whether or not S is a subspace, for $S = \{(a, b, c) : 2a - b = c\}$.

   Need to check closure under vector addition and scalar multiplication.
   VA: $(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$. We assume that $2a_1 - b_1 = c_1$ and that $2a_2 - b_2 = c_2$. Adding these we get $2a_1 - b_1 + 2a_2 - b_2 = c_1 + c_2$. Rearranging, we get $2(a_1 + a_2) - (b_1 + b_2) = (c_1 + c_2)$, hence S is closed under VA.
   SM: $d(a, b, c) = (da, db, dc)$. We assume that $2a - b = c$, multiplying by d we get $2da - db = dc$. Hence S is closed under SM, and is a subspace.

4. Consider the vector space $\mathbb{R}^2$. Show that the following set is dependent: $\{(1, 2), (2, 3), (3, 4)\}$.

   Solution 1: The dimension of $\mathbb{R}^2$ is 2, which is the maximal size of an independent set. This set must therefore be dependent.
   Solution 2: $1(1, 2) - 2(2, 3) + (3, 4) = (0, 0)$ is a nondegenerate linear combination of these vectors yielding $(0, 0)$. Other linear combinations are possible.

5. Consider the vector space $\mathbb{R}^2$. Show that the following set is spanning: $\{(1, 2), (2, 3), (3, 4)\}$.

   Given any (x, y) in $\mathbb{R}^2$, we need to find some a, b, c so that $a(1, 2) + b(2, 3) + c(3, 4) = (x, y)$.
   Many solutions are possible; for example $a = -3x + 2y, b = 2x - y, c = 0$.
   Observe that $(-3x + 2y)(1, 2) + (2x - y)(2, 3) + 0(3, 4) = (-3x + 2y, -6x + 4y) + (4x - 2y, 6x - 3y) + (0, 0) = (x, y)$.

   Note: It is not correct to claim that this set is spanning because it contains three vectors and $\mathbb{R}^2$ has dimension 2. For example, $\{(1, 0), (2, 0), (3, 0)\}$ contains three vectors, but is not spanning.